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ORIGINAL PAPER

Δ-groupoids in knot theory

R. M. Kashaev

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Abstract A Δ-groupoid is an algebraic structure which axiomatizes the combinatorics of a truncated tetrahedron. It is shown that there are relations of Δ-groupoids to rings, group pairs, and (ideal) triangulations of three-manifolds. In particular, we describe a class of representations of group pairs $H \subset G$ into the group of upper triangular two-by-two matrices over an arbitrary ring R, and associate to that group pair a universal ring so that any representation of that class factorizes through a respective ring homomorphism. These constructions are illustrated by two examples coming from knot theory, namely the trefoil and the figure-eight knots. It is also shown that one can associate a Δ-groupoid to ideal triangulations of knot complements, and a homology of Δ-groupoids is defined.

Keywords Knot theory · Ideal triangulation · Group · Malnormal subgroup · Groupoid · Ring

Mathematics Subject Classification (2000) 20L05 · 57M27 · 16S10

1 Introduction

In this paper, we introduce an algebraic structure called Δ-groupoid and describe its relationships to rings, representations of group pairs, and combinatorics of (ideal) triangulations of three-manifolds. Functorial relations of Δ-groupoids to the category of rings permit us to construct ring-valued invariants which seem to be interesting. In the case of knots, these rings are universal for a restricted class of representations of knot groups into the group $GL(2, R)$, where R is an arbitrary ring.

Ideal triangulations of link complements give rise to presentations of associated Δ-groupoids which, as groupoids with forgotten Δ-structure, have as many connected components

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R. M. Kashaev (✉)
Université de Genève, Section de mathématiques, 2–4, Rue du Lièvre, CP 64, 1211 Genève 4, Suisse
e-mail: Rinat.Kashaev@unige.ch

as the number of link components. In particular, they are connected groupoids in the case of knots. In general, two Δ-groupoids associated with two ideal triangulations of one and the same knot complement are not isomorphic, but one can argue that the corresponding vertex groups are isomorphic. In this way, we come to the most evident Δ-groupoid knot invariant to be called the *vertex group* of a knot. It is not very sensitive invariant as one can show that it is trivial one-element group in the case of the unknot, isomorphic to the group of integers $\mathbb{Z}$ for any non-trivial torus knot, and isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$ for any hyperbolic knot. Moreover, it is trivial at least for some connected sums, e.g. $3_1\#3_1$ or $3_1\#4_1$. In the light of these observations it would be interesting to calculate the vertex group for satellite knots which are not connected sums.

One can also refine the vertex group by adding extra information associated with a distinguished choice of a meridian-longitude pair. In this way one can detect the chirality of knots. For example, the torus knot $T_{p,q}$ of type (p, q) and its mirror image $T^*_{p,q}$ have isomorphic vertex groups freely generated by the meridian m, while the longitude l is given as $l = m^{pq}$ for $T_{p,q}$, and $l = m^{-pq}$ for $T^*_{p,q}$.

Finally, we define an integral Δ-groupoid homology which seems not to be very interesting in the case of hyperbolic knots, but could be of some interest in the case of non-hyperbolic knots.

The paper is organized as follows. In Sect. 2 we give a definition of the Δ-groupoid and a list of examples. In Sect. 3, we show that there is a canonical construction of a Δ-groupoid starting from a group and a malnormal subgroup. In Sect. 4, we show that two constructions in Examples 4 and 5 are functors which admit left adjoints functors. In Sects. 5 and 6, we reveal a representation theoretical interpretation of the A'-ring of the previous section in terms of a restricted class of representations of group pairs into two-by-two upper-triangular matrices with elements in arbitrary rings. In Sect. 7, in analogy with group presentations, we show that Δ-groupoids can be presented starting from tetrahedral objects. In Sect. 8, we define an integral homology of Δ-groupoids. In the construction, actions of symmetric groups in chain groups are used in an essential way.

2 Δ-groupoids

2.1 Preliminary notions and notation

Recall that a groupoid is a (small) category where all morphisms are isomorphisms [1,5]. So, a groupoid G consists of a set of objects $\mathrm{Ob}\, G$, a set of morphisms $\mathrm{Hom}(A, B)$ from A to B for any pair of objects (A, B), an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ for any object A, and a composition or product map $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ for any triple of objects (A, B, C). These data satisfy the usual category axioms with the additional condition that any morphism is invertible. Following [5], for a morphism $x \in \mathrm{Hom}(A, B)$, we write $A = \mathrm{dom}(x)$, $B = \mathrm{cod}(x)$ and call them respectively the domain (source) and the codomain (target) of x. A typical example of a groupoid is the fundamental groupoid of a topological space X, where the objects are points of X and morphisms are paths considered up to homotopies relative the end points. A group is a groupoid with one object. By analogy with group theory, we shall identify a groupoid with the union of all its morphisms, so that the composition becomes a partially defined operation. We use the convention adopted for fundamental groupoids of topological spaces, i.e. a pair of morphisms (x, y) is composable if and only if $\mathrm{cod}(x) = \mathrm{dom}(y)$, and the product is written xy rather than $y \circ x$.

For a set X, we shall write X^n for its Cartesian power, so, for example, $X^2 = X \times X$.

2.2 Definition of the Δ-groupoid

Let G be a groupoid and H its subset. We say that a pair of elements $(x, y) \in H^2$ is *H-composable* if it is composable in G and $xy \in H$.

Definition 1 A *Δ-groupoid* is a groupoid G, a generating subset $H \subset G$, and an involution $j: H \to H$, such that

(i) $i(H) = H$, where $i(x) = x^{-1}$;
(ii) the involutions i and j generate an action of the symmetric group $\mathbb{S}_3$ on the set H, i.e. the following equation is satisfied: $iji = jij$;
(iii) if $(x, y) \in H^2$ is a composable pair then $(k(x), j(y))$ is also a composable pair, where $k = iji$;
(iv) if $(x, y) \in H^2$ is H-composable then $(k(x), j(y))$ is also H-composable, and the following identity is satisfied:

$$k(xy)ik(y) = k(k(x)j(y)). \tag{1}$$

A *Δ-group* is a Δ-groupoid with one object (identity element).

A morphism between two Δ-groupoids is a groupoid morphism $f: G \to G'$ such that $f(H) \subset H'$ and $j'f = fj$. In this way, one comes to the category Δ**Gpd** of Δ-groupoids.

Remark 1 Equation (1) can be replaced by an equivalent equation of the form

$$ij(x)j(xy) = j(k(x)j(y)).$$

Remark 2 In any Δ-groupoid G there is a canonical involution $A \mapsto A^*$ acting on the set of objects (or the identities) of G. It can be defined as follows. As H is a generating set for G, for any $A \in \mathrm{Ob}G$ there exists $x \in H$ such that $A = \mathrm{dom}(x)$. We define $A^* = \mathrm{dom}(j(x))$. This definition is independent of the choice of x. Indeed, let $y \in H$ be any other element satisfying the same condition. Then, the pair $(i(y), x)$ is composable, and, therefore, so is $(ki(y), j(x))$. Thus,

$$\mathrm{dom}(j(y)) = \mathrm{cod}(ij(y)) = \mathrm{cod}(ki(y)) = \mathrm{dom}(j(x)).$$

Remark 3 Definition 1 differs from the one given in the preprint [3] in the following aspects:

(1) the subset H was not demanded to be a generating set for G so that it was possible to have empty H with non-empty G;
(2) the condition (iii), which is essential in Remark 2, was not imposed;
(3) it was implicitly assumed that any $x \in H$ enters an H-composable pair, and under that assumption the condition (ii) was superfluous.[1]

2.3 Examples of Δ-groupoids

Example 1 Let G be a group. The tree groupoid G^2 is a Δ-groupoid with $H = G^2$, $j(f, g) = (f^{-1}, f^{-1}g)$.

Example 2 Let X be a set. The set X^3 can be thought of as a disjoint sum of tree groupoids X^2 indexed by X: $X^3 \simeq \sqcup_{x\in X}\{x\} \times X^2$. In particular, $\mathrm{Ob}(X^3) = X^2$ with $\mathrm{dom}(a, b, c) = (a, b)$ and $\mathrm{cod}(a, b, c) = (a, c)$ with the product $(a, b, c)(a, c, d) = (a, b, d)$ and the inverse $(a, b, c)^{-1} = i(a, b, c) = (a, c, b)$. This is a Δ-groupoid with $H = X^3$ and $j(a, b, c) = (b, a, c)$.

[1] I am grateful to D. Bar-Natan for pointing out to this assumption during my talk at the workshop "Geometry and TQFT", Aarhus, 2007.

Example 3 We define an involution $\mathbb{Q} \cap [0, 1[\ni t \mapsto t^* \in \mathbb{Q} \cap [0, 1[$ by the following conditions: $0^* = 0$ and if $t = p/q$ with positive mutually prime integers p, q, then $t^* = \bar{p}/q$, where $\bar{p}$ is uniquely defined by the equation $p\bar{p} = -1 \pmod q$. We also define a map $\mathbb{Q} \cap [0, 1[\ni t \mapsto \hat{t} \in \mathbb{Q} \cap [0, 1]$ by the formulae $\hat{0} = 1$ and $\hat{t} = (p\bar{p} + 1)/q^2$ if $t = p/q$ with positive, mutually prime integers p, q. Notice that in the latter case $\hat{t} = \tilde{p}/q$ with $\mathbb{Z} \ni \tilde{p} = (p\bar{p} + 1)/q$, and $1 \le \tilde{p} \le \min(p, \bar{p})$. We also remark that $\widehat{t^*} = \hat{t}$.

The rational strip $X = \mathbb{Q} \times (\mathbb{Q} \cap [0, 1[)$ can be given a groupoid structure as follows. Elements (x, s) and (y, t) are composable iff $y \in s + \mathbb{Z}$, i.e. the fractional part of y is s, with the product $(x, s)(s + m, t) = (x + m, t)$, the inverse $(s + k, t)^{-1} = (t - k, s)$, and the set of units $\mathrm{Ob}X = \{(t, t)|\ t \in \mathbb{Q} \cap [0, 1[\}$. Denote by Γ_X the underlying graph of X, i.e. the subset of non-identity morphisms. One can show that X is a Δ-groupoid with $H = \Gamma_X$ and

$$k(x, t) = \left(\frac{t^* x - \hat{t}}{x - t}, t^*\right).$$

Taking into account the general construction of Sect. 3, this example is associated to the group $PSL(2, \mathbb{Z})$ and its malnormal subgroup represented by upper triangular matrices.

Example 4 Let R be a ring. We define a Δ-group AR as the subgroup of the group R^* of invertible elements of R generated by the subset $H = (1 - R^*) \cap R^*$ with $k(x) = 1 - x$ so that $j(x) = iki(x) = (1 - x^{-1})^{-1}$.

Example 5 For a ring R, let $R \rtimes R^*$ be the semidirect product of the additive group R with the multiplicative group R^* with respect to the (left) action of R^* on R by left multiplications. Set theoretically, one has $R \rtimes R^* = R \times R^*$, the group structure being given explicitly by the product $(x, y)(u, v) = (x + yu, yv)$, the unit element $(0, 1)$, and the inversion map $(x, y)^{-1} = (-y^{-1}x, y^{-1})$. We define a Δ-group BR as the subgroup of $R \rtimes R^*$ generated by the subset $H = R^* \times R^*$ with $k(x, y) = (y, x)$ so that $j(x) = kik(x, y) = (x^{-1}, -x^{-1}y)$.

Example 6 Let $(G, G_\pm, \theta)$ be a symmetrically factorized group of [4]. That means that G is a group with two isomorphic subgroups $G_\pm$ conjugated to each other by an involutive element $\theta \in G$, and the restriction of the multiplication map $m\colon G_+ \times G_- \to G_+G_- \subset G$ is a set-theoretical bijection, whose inverse is called the factorization map $G_+G_- \ni g \mapsto (g_+, g_-) \in G_+ \times G_-$. In this case, the subgroup of G_+ generated by the subset $H = G_+ \cap G_-G_+\theta \cap \theta G_+G_-$ is a Δ-group with $j(x) = (\theta x)_+$.

3 Δ-groupoids and pairs of groups

Recall that a subgroup H of a group G is called *malnormal* if the condition $gHg^{-1} \cap H \neq \{1\}$ implies that $g \in H$. In fact, for any pair of groups $H \subset G$ one can associate in a canonical way another group pair $H' \subset G'$ with malnormal H'. Namely, if N is the maximal normal subgroup of G contained in H, then we define $G' = G/N$ and $H' \subset G'$ is the malnormal closure of H/N.

Lemma 1 *Let a subgroup H of a group G be malnormal. Then, the right action of the group H^3 on the set $(G\backslash H)^2$ defined by the formula*

$$(G\backslash H)^2 \times H^3 \ni (g, h) \mapsto gh = (h_1^{-1} g_1 h_2, h_1^{-1} g_2 h_3) \in (G\backslash H)^2,$$
$$h = (h_1, h_2, h_3) \in H^3, \quad g = (g_1, g_2) \in (G\backslash H)^2$$

is free.

Proof Let $h \in H^3$ and $g \in (G\backslash H)^2$ be such that $gh = g$. On the level of components, this corresponds to two equations $h_1^{-1} g_1 h_2 = g_1$ and $h_1^{-1} g_2 h_3 = g_2$, or equivalently $g_1 h_2 g_1^{-1} = h_1$ and $g_2 h_3 g_2^{-1} = h_1$. Together with the malnormality of H these equations imply that $h_1 = h_2 = h_3 = 1$. □

We provide the set of orbits $\tilde{G} = (G\backslash H)^2/H^3$ with a groupoid structure as follows. Two orbits fH^3, gH^3 are composable iff $Hf_2H = Hg_1H$ with the product

$$fH^3 gH^3 = (f_1, f_2)H^3(g_1, g_2)H^3 = (f_1, h_0 g_2)H^3,$$

where $h_0 \in H$ is the unique element such that $f_2 H = h_0 g_1 H$. The units are $1_{HgH} = (g, g)H^3$ and the inverse of $(g_1, g_2)H^3$ is $(g_2, g_1)H^3$. Let $\Gamma(\tilde{G})$ be the underlying graph which, as a set, concides with the complement of units in $\tilde{G}$. Clearly, it is stable under the inversion. We define the map $j \colon \Gamma(\tilde{G}) \to \Gamma(\tilde{G})$ by the formula:

$$(g_1, g_2)H^3 \mapsto (g_1^{-1}, g_1^{-1} g_2)H^3.$$

Proposition 1 *The groupoid $\mathcal{G}_{G,H} = \tilde{G}$ is a Δ-groupoid with the distinguished generating subset $H = \Gamma(\tilde{G})$ and involution j.*

Proof We verify that the map j is well defined. For any $h \in H^3$ and $g \in (G\backslash H)^2$ we have

$$\begin{aligned} j(ghH^3) &= j((h_1^{-1} g_1 h_2, h_1^{-1} g_2 h_3)H^3) \\ &= (h_2^{-1} g_1^{-1} h_1, h_2^{-1} g_1^{-1} h_1 h_1^{-1} g_2 h_3)H^3 = (h_2^{-1} g_1^{-1} h_1, h_2^{-1} g_1^{-1} g_2 h_3)H^3 \\ &= (g_1^{-1}, g_1^{-1} g_2)(h_2, h_1, h_3)H^3 = (g_1^{-1}, g_1^{-1} g_2)H^3 = j(gH^3). \end{aligned}$$

Verification of the other properties is straightforward. □

Example 7 For the group $G = PSL(2, \mathbb{Z})$, let the subgroup $H \simeq \mathbb{Z}$ be given by the upper triangular matrices. One can show that H is malnormal and the associated Δ-groupoid is isomorphic to the one of Example 3. Indeed, any element $g \in G\backslash H$ is represented by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

with non-zero c, and the map $g \mapsto a/c$ is a bijection between the set of non-trivial cosets $\{gH |\ g \in G\backslash H\}$ and the set of rational numbers $\mathbb{Q}$, and the free action of H corresponds to translations by integers. Thus, the set of double cosets $\{HgH |\ g \in G\backslash H\}$ is identified with the set $\mathbb{Q}/\mathbb{Z}$. Any element of the latter has a unique representative in the semi-open unit rational interval $[0, 1[\cap\mathbb{Q}$. A morphism in the associated Δ-groupoid is given by a pair of rationals (x, y) modulo an equivalence relation given by simultaneous translations by integers, i.e a pair (x, y) is equivalent to a pair (x', y') if and only if $x - x' = y - y' \in \mathbb{Z}$. Thus, any morphism is represented by a pair (x, t), where $t \in [0, 1[\cap\mathbb{Q}$ and $x \in \mathbb{Q}$. With these identifications, calculation of structural maps is straightforward.

4 Δ-groupoids and rings

In this section we show that the constructions in Examples 4 and 5 come from functors admitting left adjoints.

Theorem 1 *The mappings* $A, B\colon \mathbf{Ring} \to \Delta\mathbf{Gpd}$ *are functors which admit left adjoints* A' *and* B'*, respectively.*

Proof The case of A. Let $f\colon R \to S$ be a morphism of rings. Then, obviously, $f(R^*) \subset S^*$. Besides, for any $x \in R$ we have $f(k(x)) = f(1-x) = f(1) - f(x) = 1 - f(x) = k(f(x))$, which implies that $f(k(R^*)) = k(f(R^*)) \subset k(S^*)$. Thus, we have a well defined morphism of Δ-groups $Af = f|_{R^*}$. If $f\colon R \to S$, $g\colon S \to T$ are two morphisms of rings, then $A(g \circ f) = g \circ f|_{R^*} = g|_{S^*} \circ f|_{R^*} = Ag \circ Af$. The proof of the functoriality of B is similar.

We define covariant functors $A', B'\colon \Delta\mathbf{Gpd} \to \mathbf{Ring}$ as follows. Let G be a Δ-groupoid. Let $\mathbb{Z}[G]$ be the groupoid ring defined similarly as for groups. Namely, it is generated by the set $\{u_x |\ x \in G\}$ with the defining relations $u_x u_y = u_{xy}$ if x and y are composable. The ring $A'G$ is the quotient ring of the groupoid ring $\mathbb{Z}[G]$ with respect to the additional relations $u_x + u_{k(x)} = 1$ for all $x \in H$. The ring $B'G$ is generated over $\mathbb{Z}$ by the elements $\{u_x, v_x |\ x \in G\}$ with the defining relations $u_x u_y = u_{xy}$, $v_{xy} = u_x v_y + v_x$ if x, y are composable, and $u_{k(x)} = v_x$, $v_{k(x)} = u_x$ for all $x \in H$. If $f \in \Delta\mathbf{Gpd}(G, M)$, we define $A'f$ and $B'f$ respectively by $A'f\colon u_x \mapsto u_{f(x)}$, and $B'f\colon u_x \mapsto u_{f(x)}, v_x \mapsto v_{f(x)}$. It is straightforward to verify the functoriality properties of these constructions.

Let us now show that A', B' are respective left adjoints of A, B. In the case of A' and A, we identify bijections $\varphi_{G,R}\colon \mathbf{Ring}(A'G, R) \simeq \Delta\mathbf{Gpd}(G, AR)$ which are natural in G and R. We define $\varphi_{G,R}(f)\colon x \mapsto f(u_x)$, whose inverse is given by $\varphi^{-1}_{G,R}(g)\colon u_x \mapsto g(x)$. Let $f \in \mathbf{Ring}(A'G, R)$. To verify the naturality in the first argument, let $h \in \Delta\mathbf{Gpd}(M, G)$. We have $A'h \in \mathbf{Ring}(A'M, A'G)$, $f \circ A'h \in \mathbf{Ring}(A'M, R)$, and $\varphi_{M,R}(f \circ A'h) \in \Delta\mathbf{Gpd}(M, AR)$ with

$$\varphi_{M,R}(f \circ A'h)\colon M \ni x \mapsto f(A'h(u_x)) = f(u_{h(x)}) = \varphi_{G,R}(f)(h(x)).$$

Thus, $\varphi_{M,R}(f \circ A'h) = \varphi_{G,R}(f) \circ h$. To verify the naturality in the second argument, let $g \in \mathbf{Ring}(R, S)$. Then,

$$\Delta\mathbf{Gpd}(G, AS) \ni \varphi_{G,S}(g \circ f)\colon x \mapsto g(f(u_x)) = g|_{R^*}(\varphi_{G,R}(f)(x)).$$

Thus, $\varphi_{G,S}(g \circ f) = Ag \circ \varphi_{G,R}(f)$. The proof in the case of B', B is similar. □

There is a natural transformation $\alpha\colon A \to B$ given by

$$\Delta\mathbf{Gpd}(AR, BR) \ni \alpha_R\colon x \mapsto (1 - x, x).$$

5 A representation theoretical interpretation of the A'-ring of pairs of groups

Let G be a group with a proper malnormal subgroup H (i.e. $H \neq G$), and $\mathcal{G}_{G,H}$, the associated Δ-groupoid described in Sect. 3. According to Theorem 1, the ring $A'\mathcal{G}_{G,H}$ is generated by the set of invertible elements $\{u_{x,y} |\ x, y \in G\backslash H\}$, which satisfy the following relations:

$$u_{x,y} u_{y,z} = u_{x,z}, \tag{2}$$

$$u_{xh,y} = u_{x,yh} = u_{hx,hy} = u_{x,y}, \quad \forall h \in H, \tag{3}$$

$$u_{y^{-1}x,y^{-1}} + u_{x,y} = 1, \quad x \neq y, \tag{4}$$

Notice that $u_{x,x} = 1$ for any $x \in G\backslash H$.

Define another ring $R_{G,H}$ generated by the set $\{s_g, v_g |\ g \in G\}$ subject to the following defining relations

$$s_1 = 1, \quad s_x s_y = s_{xy}, \quad \forall x, y \in G, \tag{5}$$

the element v_x is zero if $x \in H$ and invertible otherwise, and

$$v_{xy} = v_y + v_x s_y, \quad \forall x, y \in G. \tag{6}$$

For any ring R, let $R^* \ltimes R$ be the semidirect product of the group of invertible elements R^* and the additive group R with respect to the (right) action of R^* on R by right multiplications. As a set, the group $R^* \ltimes R$ is the product set $R^* \times R$ with the multiplication rule $(x, y)(x', y') = (xx', yx' + y')$. This construction is functorial in the sense that for any ring homomorphism $f\colon R \to S$ there corresponds a group homomorphism $\tilde{f}\colon R^* \ltimes R \to S^* \ltimes S$.

Definition 2 Given a pair of groups (G, H), where H is a malnormal subgroup of G, a ring R and a group homomorphism

$$\rho\colon G \ni x \mapsto (\alpha(x), \beta(x)) \in R^* \ltimes R.$$

We say that ρ is a *special representation* if it satisfies the following condition: for any $x \in G$, the element $\beta(x)$ is zero for $x \in H$ and invertible otherwise.

In the case of the ring $R_{G,H}$ we have the canonical special representation

$$\sigma_{G,H}\colon G \to R^*_{G,H} \ltimes R_{G,H}, \quad x \mapsto (s_x, v_x),$$

which is universal in the sense that for any ring R and any special representation $\rho\colon G \to R^* \ltimes R$ there exists a unique ring homomorphism $f\colon R_{G,H} \to R$ such that $\rho = \tilde{f} \circ \sigma_{G,H}$. The following theorem describes the representation theoretical meaning of the A'-ring.

Theorem 2 *For any $g \in G \backslash H$, the rings $A'\mathcal{G}_{G,H}$ and $R_{G,H}/(1 - v_g)$ are isomorphic.*

The proof of this theorem is split to few lemmas. Let us define a map q of the generating set of the ring $A'\mathcal{G}_{G,H}$ into the ring $R_{G,H}$ by the formula

$$q(u_{x,y}) = v_{x^{-1}} v^{-1}_{y^{-1}}.$$

Lemma 2 *The map q extends to a unique ring homomorphism $q\colon A'\mathcal{G}_{G,H} \to R_{G,H}$.*

Proof The elements $q(u_{x,y})$ are manifestly invertible and satisfy the identities $q(u_{x,z}) = q(u_{x,y})q(u_{y,z})$. The consistency of the map q with relations (3) is easily seen from the following properties of the elements v_x (which are special cases of Eq. 6):

$$v_{hx} = v_x, \quad v_{xh} = v_x s_h, \quad \forall h \in H.$$

The identity $q(u_{y^{-1}x, y^{-1}}) + q(u_{x,y}) = 1$ is equivalent to

$$v_{x^{-1}y} = v_y - v_{x^{-1}} v^{-1}_{y^{-1}} v_y$$

which, in turn, is equivalent to the defining relation (6) after taking into account the formula

$$s_y = -v^{-1}_{y^{-1}} v_y.$$

The latter formula follows from the particular case of the relation (6) corresponding to $x = y^{-1}$. $\square$

Let us fix an element $g \in G \backslash H$ and define a map f_g of the generating set of the ring $R_{G,H}$ into the ring $A'\mathcal{G}_{G,H}$ by the following formulae

$$f_g(s_x) = \begin{cases} u_{g,xg} = u_{x^{-1}g,g}, & \text{if } x \in H; \\ -u_{g,x}u_{x^{-1},g}, & \text{otherwise}, \end{cases} \tag{7}$$

and

$$f_g(v_x) = \begin{cases} 0, & \text{if } x \in H; \\ u_{x^{-1},g}, & \text{otherwise}. \end{cases} \tag{8}$$

Lemma 3 *The map f_g extends to a unique ring homomorphism $f_g \colon R_{G,H} \to A'\mathcal{G}_{G,H}$.*

Proof 1. Clearly, $f_g(1) = f_g(s_1) = u_{g,g} = 1$ and $f_g(s_x^{-1}) = f_g(s_{x^{-1}}) = f_g(s_x)^{-1}$ if $x \notin H$. For $x \in H$ we have

$$f_g(s_x^{-1}) = f_g(s_{x^{-1}}) = u_{xg,g} = u_{g,xg}^{-1} = f_g(s_x)^{-1}.$$

2. We check the identity $f_g(s_x s_y) = f_g(s_{xy}) = f_g(s_x) f_g(s_y)$ in five different cases:

$(x, y \in H)$:

$$f_g(s_{xy}) = u_{g,xyg} = u_{x^{-1}g,yg} = u_{x^{-1}g,g}u_{g,yg} = f_g(s_x) f_g(s_y);$$

$(x \in H, y \notin H)$:

$$\begin{aligned} f_g(s_{xy}) &= -u_{g,xy}u_{y^{-1}x^{-1},g} = -u_{x^{-1}g,y}u_{y^{-1},g} \\ &= -u_{x^{-1}g,g}u_{g,y}u_{y^{-1},g} = f_g(s_x) f_g(s_y); \end{aligned}$$

$(x \notin H, y \in H)$:

$$\begin{aligned} f_g(s_{xy}) &= -u_{g,xy}u_{y^{-1}x^{-1},g} = -u_{g,x}u_{x^{-1},yg} \\ &= -u_{g,x}u_{x^{-1},g}u_{g,yg} = f_g(s_x) f_g(s_y); \end{aligned}$$

$(x \notin H, y \notin H, xy \in H)$:

$$\begin{aligned} f_g(s_{xy}) = f_g(s_{xy}) f_g(s_y^{-1}) f_g(s_y) &= f_g(s_{xy}s_y^{-1}) f_g(s_y) \\ &= f_g(s_x) f_g(s_y); \end{aligned}$$

$(x \notin H, y \notin H, xy \notin H)$:

$$\begin{aligned} f_g(s_{xy}) &= -u_{g,xy}u_{y^{-1}x^{-1},g} = -u_{g,x}u_{xy,x}^{-1}u_{y^{-1}x^{-1},y^{-1}}u_{y^{-1},g} \\ &= -u_{g,x}(1 - u_{y,x^{-1}})^{-1}(1 - u_{x^{-1},y})u_{y^{-1},g} \\ &= u_{g,x}(u_{y,x^{-1}} - 1)^{-1}(u_{y,x^{-1}} - 1)u_{x^{-1},y}u_{y^{-1},g} \\ &= u_{g,x}u_{x^{-1},y}u_{y^{-1},g} = u_{g,x}u_{x^{-1}g}u_{g,y}u_{y^{-1},g} \\ &= f_g(s_x) f_g(s_y). \end{aligned}$$

3. We check the identity $f_g(v_{xy}) = f_g(v_y) + f_g(v_x) f_g(s_y)$ in five different cases:

$(x, y \in H)$: it is true trivially;
$(x \in H, y \notin H)$:

$$f_g(v_{xy}) = u_{y^{-1}x^{-1},g} = u_{y^{-1},g} = f_g(v_y);$$

$(x \notin H, y \in H)$:

$$f_g(v_{xy}) = u_{y^{-1}x^{-1},g} = u_{x^{-1},yg} = u_{x^{-1},g}u_{g,yg}$$
$$= f_g(v_x)f_g(s_y);$$

$(x \notin H, y \notin H, xy \in H)$:

$$f_g(v_y) + f_g(v_x)f_g(s_y) = u_{y^{-1},g} - u_{x^{-1},g}u_{g,y}u_{y^{-1},g}$$
$$= (1 - u_{x^{-1},y})u_{y^{-1},g} = (1 - u_{x^{-1}xy,y})u_{y^{-1},g} = 0;$$

$(x \notin H, y \notin H, xy \notin H)$:

$$f_g(v_{xy}) = u_{y^{-1}x^{-1},g} = u_{y^{-1}x^{-1},y^{-1}}u_{y^{-1},g}$$
$$= (1 - u_{x^{-1},y})u_{y^{-1},g} = (1 - u_{x^{-1},g}u_{g,y})u_{y^{-1},g}$$
$$= f_g(v_y) + f_g(v_x)f_g(s_y).$$

□

Associated with any invertible element t of the ring $R_{G,H}$ there is an endomorphism $r_t\colon R_{G,H} \to R_{G,H}$ defined on the generating elements by the formulae

$$r_t(s_x) = ts_xt^{-1}, \quad r_t(v_x) = v_xt^{-1}.$$

Note that, in general, r_t can have a non-trivial kernel, for example, if $g \in G\backslash H$, then $1 - v_g \in \ker(r_{v_g})$.

Lemma 4 *The following identities of ring homomorphisms*

$$f_g \circ q = \mathrm{id}_{A'\mathcal{G}_{G,H}}, \tag{9}$$
$$q \circ f_g = r_{v_{g^{-1}}} \tag{10}$$

hold true.

Proof Applying the left hand sides of the identities to be proved to the generating elements of the corresponding rings, we have

$$f_g(q(u_{x,y})) = f_g(v_{x^{-1}}v_{y^{-1}}^{-1}) = u_{x,g}u_{y,g}^{-1} = u_{x,y}, \quad \forall x, y \notin H,$$
$$q(f_g(s_x)) = q(u_{x^{-1}g,g}) = v_{g^{-1}x}v_{g^{-1}}^{-1} = v_{g^{-1}}s_xv_{g^{-1}}^{-1}, \quad x \in H,$$
$$q(f_g(s_x)) = q(-u_{g,x}u_{x^{-1},g}) = -v_{g^{-1}}v_{x^{-1}}^{-1}v_xv_{g^{-1}}^{-1} = v_{g^{-1}}s_xv_{g^{-1}}^{-1}, \quad \forall x \notin H,$$

and

$$q(f_g(v_x)) = q(u_{x^{-1},g}) = v_xv_{g^{-1}}^{-1}, \quad \forall x \notin H.$$

□

Lemma 5 *For any $g \in G\backslash H$, the kernel of the ring homomorphism $f_g\colon R_{G,H} \to A'\mathcal{G}_{G,H}$ is generated by the element $1 - v_{g^{-1}}$.*

Proof Let $t = v_{g^{-1}}$. Eqs. (9), (10) imply that $\ker(f_g) = \ker(r_t)$ and $r_t \circ r_t = r_t$. The latter equation means that any $x \in \ker(r_t)$ has the form $y - r_t(y)$ for some y. The identity

$$xy - r_t(xy) = (x - r_t(x))y + r_t(x)(y - r_t(y)),$$

implies that that $\ker(r_t)$ is generated by the elements $x - r_t(x)$, with x running in a generating set for the ring $R_{G,H}$. Finally, the identities

$$s_x - r_t(s_x) = s_x - ts_x t^{-1} = (1-t)s_x - ts_x t^{-1}(1-t), \quad v_x - r_t(v_x) = -v_x t^{-1}(1-t),$$

imply that $\ker(r_t)$ is generated be only one element $1 - t = 1 - v_{g^{-1}}$. □

Proof of Theorem 2 By Lemma 4, the ring homomorphism $f_g\colon R_{G,H} \to A'\mathcal{G}_{G,H}$ is surjective whose kernel, by Lemma 5, is generated by the element $1 - v_{g^{-1}}$. Thus, by the fundamental theorem in ring theory, we have an isomorphism $A'\mathcal{G}_{G,H} \simeq R_{G,H}/\ker(f_g) = R_{G,H}/(1 - v_{g^{-1}})$. But g is arbitrary element in the complement of H, and so is its inverse. Thus, by replacing g with g^{-1}, we finish the proof. □

Example 8 Consider the group pair (G, H), where

$$G = \langle a, b |\, a^2 = b^2 = 1\rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_2,$$

and $H = \langle a\rangle \simeq \mathbb{Z}_2$, a malnormal subgroup of G. Any element of the group G has the form $(ab)^m a^n$ for unique $m \in \mathbb{Z}$, $n \in \{0, 1\}$. One can show that

$$R_{G,H} \simeq \mathbb{Q}[x, x^{-1}], \quad v_{(ab)^m a^n} \mapsto (-1)^n mx, \quad s_{(ab)^m a^n} \mapsto (-1)^n,$$

while $q(u_{(ab)^m a^n,(ab)^k a^l}) \mapsto m/k$, $m, k \neq 0$. Thus, $A'\mathcal{G}_{G,H} \simeq \mathbb{Q}$.

6 Weakly special representations of group pairs

In this section, we generalize the constructions of the previous section to the case of arbitrary group pairs. In this case, we shall use a weakened version of special representations.

6.1 The group pairs (G_ρ, H_ρ)

To each group pair homomorphism of the form

$$\rho\colon (G, H) \to (R^* \ltimes R, R^*), \quad G \ni g \mapsto (\alpha(g), \beta(g)) \in R^* \ltimes R, \tag{11}$$

where R is a ring, we associate a group pair (G_ρ, H_ρ), where

$$G_\rho = G/\ker(\rho), \quad H_\rho = \beta^{-1}(0)/\ker(\rho) \subset G_\rho.$$

The set $\beta \circ \alpha^{-1}(1) \subset R$ is an additive subgroup, and the set $\alpha(G) \subset R^*$ is a multiplicative subgroup. These groups fit into the following short exact sequence of group homomorphisms

$$0 \to \beta \circ \alpha^{-1}(1) \to G_\rho \to \alpha(G) \to 1, \tag{12}$$

which sheds some light on the structure of G_ρ. In particular, there exists another short exact sequence of group homomorphisms

$$1 \to N \to G \ltimes \beta \circ \alpha^{-1}(1) \to G_\rho \to 1,$$

where the semidirect product is taken with respect to the right group action by group automorphisms

$$\beta \circ \alpha^{-1}(1) \times G \ni (x, g) \mapsto x\alpha(g) \in \beta \circ \alpha^{-1}(1),$$

and N (the image thereof) trivially intersects $\beta \circ \alpha^{-1}(1)$. We also have a group isomorphism

$$H_\rho \simeq \alpha \circ \beta^{-1}(0) \subset \alpha(G).$$

For a given group pair (G, H) the set of homomorphisms (11) is partially ordered with respect to the relation:

$$\rho < \sigma \ \Leftrightarrow\ \exists \text{ exact } (1,1) \to (N, M) \to (G_\rho, H_\rho) \to (G_\sigma, H_\sigma) \to (1,1). \tag{13}$$

6.2 The universal ring $\hat{R}_{G,H}$

Let (G, H) be a group pair. The ring $\hat{R}_{G,H}$ is generated over $\mathbb{Z}$ by the set $\{s_g, v_g |\ g \in G\}$ subject to the following defining relations:

(1) the elements s_x are invertible;
(2) the map $\sigma_{G,H}\colon G \ni x \mapsto (s_x, v_x) \in \hat{R}^*_{G,H} \ltimes \hat{R}_{G,H}$ is a group homomorphism;
(3) for any $x \in H$, $v_x = 0$.

Notice that, according to this definition, a non-zero generating element v_x is not assumed to be invertible. This ring has the following universal property: for any group pair homomorphism (11) there exists a unique ring homomorphism $f_\rho\colon \hat{R}_{G,H} \to R$ such that $\rho = \tilde{f}_\rho \circ \sigma_{G,H}$. The partial order (13) can alternatively be characterized by an equivalent condition:

$$\rho < \sigma \Leftrightarrow \ker(f_\rho) \subset \ker(f_\sigma). \tag{14}$$

6.3 Weakly special representations

Definition 3 Given a pair of groups (G, H), where H is a proper (i.e. $H \neq G$) but not necessarily malnormal subgroup of G, a nontrivial (i.e. $0 \neq 1$) ring R, and a group pair homomorphism (11). We say that ρ is a *weakly special representation* if it satisfies the following conditions: (1) $\beta(G) \subset R^* \sqcup \{0\}$; (2) $\beta(G) \neq \{0\}$.

Any special representation (if $H \neq G$) is also weakly special. For any weakly special representation ρ, the group H_ρ is a malnormal subgroup in G_ρ, and the induced representation of the pair (G_ρ, H_ρ) is special in the sense of Definition 2.

To a group pair (G, H), we associate a set valued invariant $W(G, H)$ consisting of minimal (with respect to the partial ordering (13)) weakly special representations (considered up to equivalence). Notice that if $\rho \in W(G, H)$, then the ring $A'\mathcal{G}_{G_\rho, H_\rho}$ is non-trivial (i.e. $0 \neq 1$). Taking into account characterization (14), there is a bijection between the set $W(G, H)$ and the set of ring homomorphisms f_ρ with minimal kernel.

The following proposition will be useful for calculations.

Proposition 2 *Given a group pair (G, H) with $H \neq G$, a non-trivial ring R, and a weakly special representation*

$$\rho\colon (G, H) \to (R^* \ltimes R, R^*), \quad G \ni g \mapsto (\alpha(g), \beta(g)) \in R^* \ltimes R.$$

Assume that the subring $f_\rho(\hat{R}_{G,H}) \subset R$ is generated over $\mathbb{Z}$ by the set $\alpha(G)$.

(i) *If $\beta \circ \alpha^{-1}(1) \neq \{0\}$ then $f_\rho(\hat{R}_{G,H})$ is a skew-field.*
(ii) *If $1 \in \beta \circ \alpha^{-1}(1)$, then $G_\rho \simeq \alpha(G) \ltimes \beta \circ \alpha^{-1}(1)$.*

Proof (i) We remark that $\alpha^{-1}(1)$ is a normal subgroup of G having the following properties:

$$\beta(g^{-1}tg) = \beta(t)\alpha(g), \quad \forall g \in G,\ \forall t \in \alpha^{-1}(1). \tag{15}$$

$$\beta(t_1 t_2) = \beta(t_1) + \beta(t_2), \quad \forall t_1, t_2 \in \alpha^{-1}(1). \tag{16}$$

Let $t_0 \in \alpha^{-1}(1)$ be such that $x_0 = \beta(t_0)$ is invertible. As any element $x \in f_\rho(\hat{R}_{G,H})$ can be written in the form

$$x = \sum_{i=1}^{n} m_i \alpha(g_i), \quad m_i \in \mathbb{Z},\ g_i \in G,$$

we have

$$\begin{aligned} x_0 x &= \sum_{i=1}^{n} m_i x_0 \alpha(g_i) = \sum_{i=1}^{n} m_i \beta(t_0)\alpha(g_i) = \sum_{i=1}^{n} \beta(t_0^{m_i})\alpha(g_i) \\ &= \sum_{i=1}^{n} \beta(g_i^{-1} t_0^{m_i} g_i) = \beta(\prod_{i=1}^{n} g_i^{-1} t_0^{m_i} g_i) = \beta(t_x), \quad t_x = \prod_{i=1}^{n} g_i^{-1} t_0^{m_i} g_i. \end{aligned}$$

Thus, if $x = x_0^{-1}\beta(t_x) \neq 0$, then it is invertible.

(ii) Let $t_0 \in \alpha^{-1}(1)$ be such that $\beta(t_0) = 1$. We show that the exact sequence (12) splits. Let $\xi : \alpha(G) \to G$ be a set-theoretical section to α (i.e. $\alpha \circ \xi = \mathrm{id}_{\alpha(G)}$). For any $x \in \alpha(G)$ fix a (finite) decomposition

$$\beta(\xi(x)) = \sum_i m_i(x)\alpha(f_i(x)),$$

where $m_i(x) \in \mathbb{Z}$, $f_i(x) \in G$. We define

$$\sigma : \alpha(G) \to G_\rho, \quad x \mapsto \pi_\rho\left(\xi(x)\prod_i f_i(x)^{-1} t_0^{-m_i(x)} f_i(x)\right),$$

where $\pi_\rho : G \to G_\rho$ is the canonical projection. Then, it is straightforward to see that σ is a group homomorphism such that $\alpha \circ \sigma = \mathrm{id}_{\alpha(G)}$.

□

Below, we give two examples, coming from knot theory, which indicate the fact that the set $W(G, H)$ can be an interesting and relatively tractable invariant.

Example 9 (the trefoil knot 3_1) Consider the pair of groups (G, H), where

$$G = \langle a, b |\ a^2 = b^3\rangle$$

is the fundamental group of the complement of the trefoil knot [6], and $H = \langle ab^{-1}, a^2\rangle$, the peripheral subgroup generated by the meridian $m = ab^{-1}$ and the longitude $l = a^2(ab^{-1})^{-6}$. As the element a^2 is central, the subgroup H is not malnormal in G. If we take the quotient group with respect to the center

$$G/\langle a^2\rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL(2, \mathbb{Z}),$$

then the image of the subgroup $H/\langle a^2\rangle \simeq \mathbb{Z}$ is malnormal, and it is identified with the subgroup of upper triangular matrices in $PSL(2, \mathbb{Z})$. Thus, one can construct the Δ-groupoid $\mathcal{G}_{\tilde{G},\tilde{H}}$ (which is isomorphic to the one of Example 3), but the corresponding A'-ring happens to be trivial (i.e. $0 = 1$). This is a consequence of the result to be proved below: the set $W(G, H)$ consists of a single element ρ_0 such that, the A'-ring of the pair (G_{ρ_0}, H_{ρ_0}) is isomorphic to the field $\mathbb{Q}[t]/(\Delta_{3_1}(t))$, where $\Delta_{3_1}(t) = t^2 - t + 1$ is the Alexander polynomial of the trefoil knot.

The ring $\hat{R}_{G,H}$ admits the following finite presentation: it is generated over $\mathbb{Z}$ by four elements s_a, s_b, v_a, v_b, of which s_a, s_b are invertible, subject to four relations

$$s_a^2 = s_b^3, \quad v_a = v_b, \quad v_a(1+s_a) = v_b(1+s_b+s_b^2) = 0.$$

We consider a ring homomorphism

$$\phi\colon \hat{R}_{G,H} \to \mathbb{Z}[t]/(\Delta_{3_1}(t)), \quad s_a \mapsto -1, \quad s_b \mapsto -t^{-1}, \quad v_a \mapsto 1, \quad v_b \mapsto 1.$$

Theorem 3 (i) *For any weakly special representation ρ' there exists an equivalent representation ρ such that the ring homomorphism f_ρ factorizes through ϕ, i.e. there exists a unique ring homomorphism $h_\rho\colon \mathbb{Z}[t]/(\Delta_{3_1}(t)) \to R$ such that $f_\rho = h_\rho \circ \phi$.*

(ii) *The kernel of the group homomorphism $\tilde{\phi} \circ \sigma_{G,H}$ is generated by a^2 and $(ab^{-1})^6$ with the quotient group pair $(\tilde{G}, \tilde{H})$,*

$$\tilde{G} = \langle a, b |\, a^2 = b^3 = (ab^{-1})^6 = 1\rangle, \quad \tilde{H} = \langle ab^{-1}\rangle,$$

where $\tilde{H}$ is malnormal in $\tilde{G}$.

(iii) *The ring $A'\mathcal{G}_{\tilde{G},\tilde{H}}$ is isomorphic to the field $\mathbb{Q}[t]/(\Delta_{3_1}(t))$.*

Proof (i) Let R be a non-trivial ring and

$$\rho\colon G \ni g \mapsto (\alpha(g), \beta(g)) \in R^* \ltimes R,$$

a weakly special representation. We have

$$\beta(a) = \beta(ab^{-1}b) = \beta(ab^{-1})\alpha(b) + \beta(b) = \beta(b),$$

and

$$0 = \beta(a^2) = \beta(a)(\alpha(a)+1), \quad 0 = \beta(b^3) = \beta(b)(\alpha(b)^2 + \alpha(b) + 1).$$

The element $\xi = \beta(a) = \beta(b)$ is invertible, since otherwise $\xi = 0$ and $\beta^{-1}(0) = G$. Thus, $\alpha(a) = -1$ and $\alpha(b)$ is an element satisfying the equation $\Delta_{3_1}(-\alpha(b)^{-1}) = 0$. Replacing ρ by an equivalent representation, we can assume that $\xi = 1$.
The ring homomorphism f_ρ is defined by the images of the generating elements

$$s_a \mapsto -1, \quad s_b \mapsto \alpha(b), \quad v_a \mapsto 1, \quad v_b \mapsto 1,$$

and it is easy to see that we have a factorization $f_\rho = h_\rho \circ \phi$, with a unique ring homomorphism

$$h_\rho\colon \mathbb{Z}[t]/(\Delta_{3_1}(t)) \to R, \quad t \mapsto -\alpha(b)^{-1}.$$

(ii) It is easily verified that $a^2, (ab^{-1})^6 \in \ker(\tilde{\phi} \circ \sigma_{G,H})$. We remark an isomorphism

$$\tilde{G} \simeq \langle s, t_1, t_2 |\, s^6 = 1,\ t_1t_2 = t_2t_1,\ st_1 = t_2s,\ t_1st_2 = t_2s\rangle \simeq \mathbb{Z}_6 \ltimes \mathbb{Z}^2$$

given, for example, by the formulae:

$$s \mapsto ab^{-1}, \quad t_1 \mapsto babab, \quad t_2 \mapsto bab^{-1}a,$$

and the induced from $\tilde{\phi} \circ \sigma_{G,H}$ group homomorphism takes the form

$$s \mapsto (t, 0), \quad t_1 \mapsto (1, 1), \quad t_2 \mapsto (1, t^{-1}),$$

so that a generic element $t_1^m t_2^n s^k$, $m, n \in \mathbb{Z}$, $k \in \mathbb{Z}_6$, has the image $(t^k, (m + nt^{-1})t^k)$. The latter is the identity element $(1, 0)$ if and only if $k = 0 \pmod 6$ and $m = n = 0$.

(iii) The pair (G, H) and any weakly special representation ρ satisfy the conditions of Proposition 2, and thus the ring $A'\mathcal{G}_{G_\rho,H_\rho}$ is the localization of a homomorphic image of the ring $\mathbb{Z}[t]/(\Delta_{3_1}(t))$ at all non-zero elements. The ring $\mathbb{Z}[t]/(\Delta_{3_1}(t))$ itself is a commutative integral domain, and thus its minimal quotient ring corresponds to the zero ideal. In this way, we come to the field $\mathbb{Q}[t]/(\Delta_{3_1}(t))$ which is the A'-ring associated to the only minimal weakly special representation ρ_0 with $(G_{\rho_0}, H_{\rho_0}) = (\tilde{G}, \tilde{H})$. □

Corollary 1 *The set $W(G, H)$ is a singleton consisting of a minimal weakly special representation ρ_0 such that $H \cap \ker(\rho_0) = \langle m^6, l\rangle$.*

Remark 4 The group pair $(\tilde{G}, \tilde{H})$ is the quotient of the pair (G, H) with respect to the normal subgroup of G generated by the center of G and the longitude $l = a^2(ab^{-1})^{-6}$.

Example 10 (the figure-eight knot 4_1) The fundamental group of the complement admits the following presentation [6]:

$$G = \langle a_1, a_2 | \, a_1 w_a = w_a a_2\rangle, \quad w_a = a_2^{-1} a_1 a_2 a_1^{-1},$$

the peripheral subgroup being given by

$$H = \langle m = a_1, l = w_a \bar{w}_a\rangle \simeq \mathbb{Z}^2, \quad \bar{w}_a = a_1^{-1} a_2 a_1 a_2^{-1}.$$

One can show that H is a malnormal subgroup of G (this is true for any hyperbolic knot), and that the corresponding A'-ring is trivial. The latter fact will follow from our description of the set $W(G, H)$: it consists of two minimal weakly special representations ρ_i, $i \in \{1, 2\}$, such that

$$H \cap \ker(\rho_i) = \langle m^{p_i}, l m^{q_i}\rangle, \quad (p_1, q_1) = (0, 0), \quad (p_2, q_2) = (6, 3).$$

The ring $\hat{R}_{G,H}$ admits the following finite presentation: it is generated by four elements $\{s_{a_i}, v_{a_i} | \, i \in \{1, 2\}\}$, with s_{a_i} being invertible, subject to four relations:

$$s_{a_1} s_{w_a} = s_{w_a} s_{a_2}, \quad v_{a_1} = 0, \quad v_{\bar{w}_a} + v_{w_a} s_{\bar{w}_a} = 0, \quad v_{w_a}(1 - s_{a_2}) = v_{a_2},$$

where

$$s_{w_a} = s_{a_2}^{-1} s_{a_1} s_{a_2} s_{a_1}^{-1}, \quad s_{\bar{w}_a} = s_{a_1}^{-1} s_{a_2} s_{a_1} s_{a_2}^{-1},$$
$$v_{w_a} = v_{a_2}(s_{a_1}^{-1} - s_{w_a}), \quad v_{\bar{w}_a} = -v_{a_2}(1 - s_{a_1}) s_{a_2}^{-1}.$$

We also define a ring S generated over $\mathbb{Z}$ by three elements $\{p, r, x\}$, and the following defining relations:

$$x^2 = x - p, \tag{17}$$
$$px = x + 3p + r^2 - 1, \tag{18}$$
$$pr = r, \tag{19}$$
$$rx + xr = r - r^2, \tag{20}$$
$$p^2 = 1 - 4p - 2r^2. \tag{21}$$

We remark that this ring is noncommutative and finite dimensional over $\mathbb{Z}$ with $\dim_{\mathbb{Z}} S = 6$. Indeed, it is straightforward to see that for a $\mathbb{Z}$-linear basis one can choose, for example, the set $\{1, p, r, x, rx, r^2\}$, the set $\{1, p, r^2\}$ being a $\mathbb{Z}$-basis of the center.

Lemma 6 *In the ring S, let (a) be the two sided ideal generated by the element*

$$a = 2 + 2p + r + 2x + 3r^2 + rx.$$

Then, $\{2, r, 1 - p\} \subset (a)$.

Proof First, one can verify that $2 + r = ba$, where

$$b = 1 - x - r^2 - 2rx,$$

so that $2 + r \in (a)$. Next, one has $r = ca + (2 + r)x$, where

$$c = -2 + p + 3r + 4x - rx,$$

so that $r \in (a)$, and thus $2 \in (a)$. Finally, remarking that

$$1 - p = 2x + r^2 x,$$

we conclude that $1 - p \in (a)$. □

Lemma 7 *Let p, q, x be three elements of a ring R satisfying the three identities*

$$p = x - x^2, \tag{22}$$
$$q = px - 3p - x + 1, \tag{23}$$
$$pq = q. \tag{24}$$

Then, the element p is invertible if and only if

$$2q + p^2 + 4p - 1 = 0. \tag{25}$$

Proof First, multiplying by q identity (23) and simplifying the right hand side by the use of Eq. (24), we see that

$$q^2 = -2q. \tag{26}$$

Next, excluding x from identities (22) and (23), we obtain the identity

$$q^2 + (5p - 1)q + p(p^2 + 4p - 1) = 0$$

which, due to Eqs. (24) and (26), simplifies to

$$2q + p(p^2 + 4p - 1) = 0. \tag{27}$$

Now, if p is invertible, then Eqs. (24), (27) imply Eq. (25). Conversely, if (25) is true, then combining it with (27), we obtain the polynomial identity

$$(p - 1)(p^2 + 4p - 1) = 0$$

which implies invertibility of p with the inverse $p^{-1} = 5 - 3p - p^2$. □

Theorem 4 (i) *There exists a unique ring homomorphism $\phi\colon \hat{R}_{G,H} \to S$ such that*

$$s_{a_1} \mapsto x, \quad s_{a_2} \mapsto x + r, \quad v_{a_1} \mapsto 0, \quad v_{a_2} \mapsto 1.$$

(ii) *For any weakly special representation ρ, considered up to equivalence, the ring homomorphism f_ρ factorizes through ϕ, i.e. there exists a unique ring homomorphism $h_\rho\colon S \to R$ such that $f_\rho = h_\rho \circ \phi$.*

(iii) *The set $W(G, H)$ consists of two elements.*

Proof (i) This is a straightforward verification.

(ii) Let R be a non-trivial ring and

$$\rho\colon G \ni g \mapsto (\alpha(g), \beta(g)) \in R^* \ltimes R,$$

a weakly special representation of G. Then, the element $\xi = \beta(a_2)$ is invertible, since otherwise $\beta^{-1}(0) = G$. By replacing ρ with an equivalent representation, we can assume that $\xi = 1$.

Denote

$$x_i = \alpha(a_i), \quad w_x = \alpha(w_a), \quad \bar{w}_x = \alpha(\bar{w}_a).$$

Note that

$$0 = \beta(a_2 a_2^{-1}) = \beta(a_2^{-1}) + x_2^{-1} \Leftrightarrow \beta(a_2^{-1}) = -x_2^{-1}.$$

From the definitions of w_a and $\bar{w}_a$ we have

$$\beta(w_a) = \beta(a_2^{-1} a_1 a_2 a_1^{-1}) = \beta(a_1 a_2 a_1^{-1}) + \beta(a_2^{-1}) x_1 x_2 x_1^{-1} = x_1^{-1} - w_x, \tag{28}$$

$$\beta(\bar{w}_a) = \beta(a_1^{-1} a_2 a_1 a_2^{-1}) = -x_2^{-1} + x_1 x_2^{-1} = -(1 - x_1) x_2^{-1}. \tag{29}$$

From the relation $a_1 w_a = w_a a_2$ we obtain an identity

$$\beta(w_a) = \beta(a_1 w_a) = \beta(w_a a_2) = \beta(w_a) x_2 + 1$$

which implies that $\beta(w_a) \neq 0$, and thus $\beta(w_a)$ is invertible with

$$\beta(w_a)^{-1} = 1 - x_2.$$

Invertibility of $\beta(\bar{w}_a)$ follows from the equation

$$0 = \beta(w_a \bar{w}_a) = \beta(w_a) \bar{w}_x + \beta(\bar{w}_a).$$

Compatibility of the latter equations with the formulae (28), (29) implies the following anticommutation relations

$$x_i^{-1} x_j + x_j x_i^{-1} = x_i^{-1} + x_j - 1, \quad \{i, j\} = \{1, 2\}. \tag{30}$$

Indeed,

$$\begin{aligned} x_1 x_2^{-1} &= (1 - x_2)\beta(w_a) x_1 x_2^{-1} = (1 - x_2)(x_1^{-1} - w_x) x_1 x_2^{-1} \\ &= (1 - x_2)(x_1^{-1} - x_2^{-1} x_1 x_2 x_1^{-1}) x_1 x_2^{-1} = (1 - x_2)(x_2^{-1} - x_2^{-1} x_1) \\ &= (x_2^{-1} - 1)(1 - x_1) = x_1 + x_2^{-1} - 1 - x_2^{-1} x_1, \end{aligned}$$

and

$$\begin{aligned} x_1^{-1} x_2 &= x_1^{-1} x_2 x_1 x_2^{-1} x_2 x_1^{-1} = \bar{w}_x x_2 x_1^{-1} = -\beta(w_a)^{-1} \beta(\bar{w}_a) x_2 x_1^{-1} \\ &= (1 - x_2)(1 - x_1) x_2^{-1} x_2 x_1^{-1} = (1 - x_2)(x_1^{-1} - 1) = x_2 + x_1^{-1} - 1 - x_2 x_1^{-1}. \end{aligned}$$

By using relations (30), one obtains the following formula:

$$w_x^{-1} = (x_2^{-1} - 1)(1 - x_1 - x_2) x_1^{-1}.$$

Indeed,

$$\begin{aligned} w_x^{-1} &= x_1 x_2^{-1} x_1^{-1} x_2 = (x_1 + x_2^{-1} - 1 - x_2^{-1} x_1) x_1^{-1} x_2 = (x_2^{-1} - 1)(1 - x_1) x_1^{-1} x_2 \\ &= (x_2^{-1} - 1)(x_1^{-1} x_2 - x_2) = (x_2^{-1} - 1)(x_1^{-1} - 1 - x_2 x_1^{-1}) \\ &= (x_2^{-1} - 1)(1 - x_1 - x_2) x_1^{-1}. \end{aligned}$$

This formula implies the following equivalences:

$$x_2 w_x^{-1} = w_x^{-1} x_1 \;\Leftrightarrow\; x_2(1 - x_1 - x_2) = (1 - x_1 - x_2)x_1 \;\Leftrightarrow\; x_2(1 - x_2) = (1 - x_1)x_1.$$

The latter identity can be equivalently rewritten in the form

$$x_1 x_{21} + x_{21} x_1 = x_{21} - x_{21}^2, \tag{31}$$

where

$$x_{21} = x_2 - x_1,$$

and the same identity implies that there exists an invertible element z such that

$$x_i(1 - x_i) = z, \quad i \in \{1, 2\}.$$

Evidently, z commutes with both x_1 and x_2. We have the following formulae for the inverses of x_i:

$$x_i^{-1} = (1 - x_i)z^{-1},$$

substitute them into Eq. (30), and rewrite the result as the following two equations

$$x_1 x_{21} + x_{21} x_1 = x_{21} + 3z + x_1 - z x_1 - 1, \tag{32}$$

and

$$(z - 1)x_{21} = 0. \tag{33}$$

Compatibility of Eqs. (31) and (32) gives one more identity

$$z x_1 + 1 = 3z + x_1 + x_{21}^2. \tag{34}$$

Finally, applying Lemma 7 to elements p, x_{21}^2, x_1, we obtain one more identity

$$2x_{21}^2 + p^2 + 4p - 1 = 0.$$

Now, we can easily see that the mapping

$$h_\rho(p) = z, \quad h_\rho(r) = x_{21}, \quad h_\rho(x) = x_1,$$

in a unique way extends to a ring homomorphism $h_\rho \colon S \to R$, and one has the factorization formula $f_\rho = h_\rho \circ \phi$.

(iii) We remark that our pair (G, H) and any weakly special representation ρ verify the conditions of Proposition 2. For example, let us check that there exists an element $t_0 \in \alpha^{-1}(1)$ such that $\beta(t_0)$ is invertible. In the case if $x_{21} = 0$, we can choose $t_0 = a_1^{-1} a_2$ for which $\beta(t_0) = 1$, while for $x_{21} \neq 0$ we can choose $t_0 = w_a \bar{w}_a^{-1}$ for which

$$\beta(t_0) = 2 + 2z + x_{21} + 2x_1 + 3x_{21}^2 + x_{21} x_1.$$

Due to Lemma 6, the latter element is non-zero and therefore invertible. Thus, taking into account the parts (i) and (ii), to identify the elements of the set $W(G, H)$, it is enough to find all minimal quotients of the ring S which can be embedded into rings the way that all non-zero elements become invertible. In particular, the quotient rings must be integral domains.

We have two relations in S which indicate existence of zero-divisors:

$$r(1-p)=0, \quad r(2+r^2)=0.$$

We have the following mutually excluding possibilities for minimal quotients which remove these relations:

(1) $r=0$;
(2) $p=1, r^2=-2\neq 0$.

The case (1) gives the quotient ring

$$S/(r)\simeq \mathbb{Z}[t]/(\Delta_{4_1}(t)), \quad \Delta_{4_1}(t)=t^2-3t+1, \quad x\mapsto t,\ p\mapsto 1-2t.$$

This is a commutative integral domain, and its localization at the set of non-zero elements coincides with the field of fractions $\mathbb{Q}[t]/(\Delta_{4_1}(t))$.
The case (2) gives the quotient ring

$$S/(1-p,2+r^2)\simeq \mathbb{Z}\langle r,x\rangle/(1-x+x^2, rx+xr-2-r),$$

which is isomorphic to the ring of Hurwitz integral quaternions

$$H=\left\{a+bi+cj+dk|\ (a,b,c,d)\in \mathbb{Z}^4\sqcup(2^{-1}+\mathbb{Z})^4\right\},$$

where

$$1+i^2=1+j^2=ij+ji=0, \quad k=ij,$$

the isomorphism being given by the map

$$r\mapsto i+j, \quad x\mapsto 2^{-1}(1-i-j+k),$$

and the inverse map

$$i\mapsto rx-1, \quad j\mapsto xr-1.$$

Thus, $S/(1-p,2+r^2)$ admits an embedding into a (non-commutative) division ring of rational quaternions.

□

Let $\rho_i, i\in\{1,2\}$, represent the two elements of $W(G,H)$. Corresponding to ρ_1 the group pair (G_1,H_1) admits a presentation

$$G_1=\langle s,t_0,t_1|\ t_0t_1=t_1t_0,\ st_1=t_0s,\ t_1st_0=t_0^3s\rangle\simeq\mathbb{Z}\ltimes\mathbb{Z}^2, \quad H_1=\langle s\rangle\simeq\mathbb{Z}$$

with the projection homomorphism

$$\pi_1\colon (G,H)\to(G_1,H_1), \quad a_1\mapsto s,\ a_2\mapsto st_0,$$

whose kernel $\ker(\pi_1)$ can be shown to be generated by the longitude $l=w_a\bar{w}_a$. In other words, we have a group isomorphism

$$G_1\simeq\langle a_1,a_2|\ a_1w_a=w_aa_2,\ w_a\bar{w}_a=1\rangle, \quad w_a=a_2^{-1}a_1a_2a_1^{-1},\ \bar{w}_a=a_1^{-1}a_2a_1a_2^{-1}.$$

To describe the group pair (G_2,H_2), corresponding to ρ_2, consider the following representation of G in $SL(4,\mathbb{Z})$:

$$a_1\mapsto\begin{pmatrix}0&1&0&0\\-1&1&0&0\\0&0&0&1\\0&0&-1&1\end{pmatrix}, \quad a_2\mapsto\begin{pmatrix}0&0&1&0\\0&1&1&-1\\-1&0&1&0\\-1&1&1&0\end{pmatrix},$$

whose kernel is generated, for example, by the element $a_1^{-1}a_2a_1^2a_2$, and the corresponding right action of G on $\mathbb{Z}^4$ (given by the multiplication of integer row vectors by above matrices). The group G_2 is the quotient group of the semidirect product $G \ltimes \mathbb{Z}^4$ by the relation $a_1(1,0,0,0) = a_2a_1^2a_2$, where we identify G and $\mathbb{Z}^4$ as subgroups of $G \ltimes \mathbb{Z}^4$, while the subgroup $H_2 = \langle a_1 \rangle \simeq \mathbb{Z}_6$.

7 Presentations of Δ-groupoids

7.1 The tetrahedral category

For any non-negative integer $n \geq 0$ we identify the symmetric group $\mathbb{S}_n$ as the sub-group of all permutations (bijections) of the set of non-negative integers $\mathbb{Z}_{\geq 0}$ acting identically on the subset $\mathbb{Z}_{\geq n} \subset \mathbb{Z}_{\geq 0}$. This interpretation fixes a canonical inclusion $\mathbb{S}_m \subset \mathbb{S}_n$ for any pair $m \leq n$. The standard generating set $\{s_i = (i-1,i) | 1 \leq i < n\}$ of $\mathbb{S}_n$ is given by elementary transpositions of two consecutive integers. Later on, it will be convenient to use the inductive limit

$$\mathbb{S}_\infty = \varinjlim \mathbb{S}_n = \cup_{n \geq 0} \mathbb{S}_n.$$

We also denote by $\mathbb{S}_n\mathbf{Set}$ the category of $\mathbb{S}_n$-sets, i.e. sets with a left $\mathbb{S}_n$-action and $\mathbb{S}_n$-equivariant maps as morphisms.

We remark that in any Δ-groupoid G, its distinguished generating set H is an $\mathbb{S}_3$-set given by the identifications $s_1 = j$ and $s_2 = i$, while the set $V \subset H^2$ of H-composable pairs is an $\mathbb{S}_4$-set given by the rules

$$s_1(x,y) = (j(x), j(k(x)j(y))), \quad s_2(x,y) = (i(x), xy), \quad s_3(x,y) = (xy, i(y)), \tag{35}$$

and the projection map to the first component

$$\mathrm{pr}_1 \colon V \ni (x,y) \mapsto x \in H$$

being $\mathbb{S}_3$-equivariant.

Let $R_{43} \colon \mathbb{S}_4\mathbf{Set} \to \mathbb{S}_3\mathbf{Set}$ be the restriction (to the subgroup) functor. Consider the comma category[2] $(R_{43} \downarrow \mathbb{S}_3\mathbf{Set})$ of $\mathbb{S}_3$-equivariant maps $a \colon R_{43}(V_a) \to I_a$, for some $\mathbb{S}_4$-set V_a, and $\mathbb{S}_3$-set I_a. Call it the *tetrahedral* category. An object of this category will be called *tetrahedral object*. A morphism between two tetrahedral objects $f \colon a \to b$, called *tetrahedral morphism*, is a pair (p_f, q_f) where $p_f \in \mathbb{S}_4\mathbf{Set}(V_a, V_b)$ and $q_f \in \mathbb{S}_3\mathbf{Set}(I_a, I_b)$ are such that $bR_{43}(p_f) = q_f a$. Taking into account the remarks above, to each Δ-groupoid G, we can associate a tetrahedral object $C(G) = \mathrm{pr}_1 \colon R_{43}(V) \to H$. If $f \colon G \to G'$ is a morphism of Δ-groupoids, then the pair

$$C(f) = (f \times f|_V, f|_H)$$

is a tetrahedral morphism such that $C(fg) = C(f)C(g)$. Thus, we obtain a functor $C \colon \Delta\mathbf{Gpd} \to (R_{43} \downarrow \mathbb{S}_3\mathbf{Set})$.

A Δ-groupoid G is called *finite* if its distinguished set H is finite (note that G itself can be an infinite groupoid). Let $\Delta\mathbf{Gpd}_{\mathrm{fin}}$ be the full subcategory of finite Δ-groupoids, and $(R_{43} \downarrow \mathbb{S}_3\mathbf{Set})_{\mathrm{fin}}$ the full subcategory of *finite* tetrahedral objects (i.e. a's with finite V_a and I_a). Then, the functor C restricts to a functor

$$C_{\mathrm{fin}} \colon \Delta\mathbf{Gpd}_{\mathrm{fin}} \to (R_{43} \downarrow \mathbb{S}_3\mathbf{Set})_{\mathrm{fin}}.$$

[2] see [5] for a general definition of a comma category.

Theorem 5 *The functor C_{fin} admits a left adjoint*

$$C'_{\text{fin}}\colon (R_{43}\downarrow \mathbb{S}_3\mathbf{Set})_{\text{fin}} \to \Delta\mathbf{Gpd}_{\text{fin}}$$

which verifies the identities

$$C'_{\text{fin}}C_{\text{fin}}C'_{\text{fin}} = C'_{\text{fin}}, \quad C_{\text{fin}}C'_{\text{fin}}C_{\text{fin}} = C_{\text{fin}}.$$

Proof Let a be an arbitrary finite tetrahedral object. Consider a map

$$\tau_a\colon V_a \to I_a^2, \quad \tau_a(v) = (a(v), a((321)(v))), \quad \forall v \in V_a$$

Let $\mathcal{R}$ be the minimal $\mathbb{S}_4$-equivariant equivalence relation on V_a generated by the set

$$\bigcup_{x\in I_a^2} \tau_a^{-1}(x)^2$$

and $\mathcal{R}^a$, the a-image of $\mathcal{R}$ which is necessarily an $\mathbb{S}_3$-equivariant equivalence relation on I_a. Let $p_a\colon V_a \to V_a/\mathcal{R}$ and $q_a\colon I_a \to I_a/\mathcal{R}^a$ be the canonical equivariant projections on the quotient sets. There exists a unique tetrahedral object a_1 such that the pair (p_a, q_a) is a tetrahedral morphism from a to a_1. Iterating this procedure, we obtain a sequence of tetrahedral morphisms

$$a \to a_1 \to a_2 \to \cdots$$

which, due to finiteness of a, stabilizes in a finite number of steps to a tetrahedral object $\tilde{a}$. It is characterized by the property that the map $\tau_{\tilde{a}}$ is an injection, so that we can identify the set $V_{\tilde{a}}$ with its $\tau_{\tilde{a}}$-image in $I_{\tilde{a}}^2$.

For any $(x, y) \in V_{\tilde{a}}$ denote $\tilde{a}(s_3(x, y)) = xy$ and call it *product*. Then, the action of the group $\mathbb{S}_4$ on $V_{\tilde{a}}$ is given by the formulae (35), where $i(x) = s_2(x)$, $j(x) = s_1(x)$, $k(x) = (02)(x)$, and the consistency conditions imply the following properties of the product:

$$i(xy) = i(y)i(x), \quad k(xy) = k(k(x)j(y))k(y), \quad i(x)(xy) = (yx)i(x) = y. \tag{36}$$

However, this product is not necessarily associative, and to repair that, we consider the minimal $\mathbb{S}_3$-equivariant equivalence relation $\mathcal{S}$ on $I_{\tilde{a}}$ generated by the relations $x(yz) \sim (xy)z$, where all products are supposed to make sense. Let $\pi_{\tilde{a}}\colon I_{\tilde{a}} \to I_{\tilde{a}}/\mathcal{S}$ be the canonical $\mathbb{S}_3$-equivariant projection on the quotient set. Then, clearly, $\tilde{a}' = \pi_{\tilde{a}} \circ \tilde{a}$ is another (finite) tetrahedral object with $V_{\tilde{a}'} = V_{\tilde{a}}$ and $I_{\tilde{a}'} = I_{\tilde{a}}/\mathcal{S}$, and with canonically associated tetrahedral morphism $(\mathrm{id}_{\tilde{a}}, \pi_{\tilde{a}})\colon \tilde{a} \to \tilde{a}'$. Applying the "tilde" operation to $\tilde{a}'$, we obtain a composed morphism

$$(p_{\tilde{a}'}, q_{\tilde{a}'}) \circ (\mathrm{id}_{\tilde{a}}, \pi_{\tilde{a}}) \circ (p_a, q_a)\colon a \to \hat{a} = \widetilde{\tilde{a}'}$$

Again, the iterated sequence of such morphisms

$$a \to \hat{a} \to \hat{\hat{a}} \to \cdots$$

stabilizes in finite number of steps to an object $\dot{a}$ with $V_{\dot{a}} \subset I_{\dot{a}}^2$ and the associative product $xy = \dot{a}(s_3(x, y))$ satisfying the relations (36).

Now, let $\mathcal{T}$ be the minimal equivalence relation on $I_{\dot{a}}$ generated by the set $(i \times \mathrm{id}_{I_{\dot{a}}})(V_{\dot{a}})$. Denote by

$$\mathrm{dom}\colon I_{\dot{a}} \to N_{\dot{a}} = I_{\dot{a}}/\mathcal{T}$$

the canonical projection to the quotient set. Define another map

$$\mathrm{cod} = \mathrm{dom} \circ i : I_{\dot{a}} \to N_{\dot{a}}.$$

In this way, we obtain a graph (quiver) Γ_a with the set of arrows $I_{\dot{a}}$, the set of nodes $N_{\dot{a}}$, and the domain (source) and the codomain (target) functions $\mathrm{dom}(x)$, $\mathrm{cod}(x)$. Thus, we obtain a finite Δ-groupoid with the presentation

$$C'_{\mathrm{fin}}(a) = \langle \Gamma_a | \; x \circ y = xy \text{ if } (x, y) \in V_{\dot{a}} \rangle$$

whose distinguished generating set is given by $I_{\dot{a}}$. □

7.2 Δ-complexes

Let

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in [0, 1]^{n+1} | \; t_0 + t_1 + \cdots + t_n = 1\}, \quad n \geq 0$$

be the standard n-simplex with face inclusion maps

$$\delta_m : \Delta^n \to \Delta^{n+1}, \quad 0 \leq m \leq n + 1$$

defined by

$$\delta_m(t_0, \ldots, t_n) = (t_0, \ldots, t_{m-1}, 0, t_m, \ldots, t_n)$$

A (simplicial) cell in a topological space X is a continuous map

$$f : \Delta^n \to X$$

such that the restriction of f to the interior of Δ^n is an embedding. On the set of cells $\Sigma(X)$ we have the dimension function

$$d : \Sigma(X) \to \mathbb{Z}, \quad (f : \Delta^n \to X) \mapsto n.$$

Hatcher in [2] introduces Δ-complexes as a generalization of simplicial complexes. A Δ-complex structure on a topological space X can be defined as a pair $(\Delta(X), \partial)$, where $\Delta(X) \subset \Sigma(X)$ and ∂ is a set of maps

$$\partial = \left\{ \partial_n : d|^{-1}_{\Delta(X)}(\mathbb{Z}_{\geq \max(1,n)}) \to d|^{-1}_{\Delta(X)}(\mathbb{Z}_{\geq n-1}) \;\middle|\; n \geq 0 \right\}$$

such that:

(i) each point of X is in the image of exactly one restriction, $\alpha|_{\overset{\circ}{\Delta}{}^n}$ for $\alpha \in \Delta(X)$;
(ii) $\alpha \circ \delta_m = \partial_m \alpha$;
(iii) a set $A \subset X$ is open iff $\alpha^{-1}(A)$ is open for each $\alpha \in \Delta(X)$.

Clearly, any Δ-complex is a CW-complex.

7.3 Tetrahedral objects from Δ-complexes

We associate a tetrahedral object a_X to a Δ-complex X as follows:

$$V_{a_X} = \mathbb{S}_4 \times \Delta^3(X), \quad I_{a_X} = \mathbb{S}_3 \times \Delta^2(X),$$

which are $\mathbb{S}_n$-sets with the groups acting by left multiplications on the first components, and

$$a_X((i3), x) = (g_i, \partial_i x), \quad i \in \{0, 1, 2, 3\}, \quad g_0 = (012), \quad g_1 = (12), \quad g_2 = g_3 = 1,$$

where $(33) = 1$, and the value $a_X(g, x)$ for $g \neq (i3)$ is uniquely deduced from its $\mathbb{S}_3$-equivariance property.

7.4 Ideal triangulations of knot complements

A particular class of three dimensional Δ-complexes arises as ideal triangulations of knot complements. A simple caculation shows that in any ideal triangulation there are equal number of edges and tetrahedra and twice as many faces. For example, the two simplest non-trivial knots (the trefoil and the figure-eight) admit ideal triangulations with only two tetrahedra $\{u, v\}$, four faces $\{a, b, c, d\}$ and two edges $\{p, q\}$, the difference being in the gluing rules. Using the notation $(x|\partial_0 x, \partial_1 x, \dots, \partial_n x)$, these examples read as follows.

Example 11 (The trefoil knot) The gluing rules are given by a list

$$(u|a, b, c, d),\ (v|d, c, b, a),\ (a|p, p, p),\ (b|p, q, p),\ (c|p, q, p),\ (d|p, p, p).$$

The associated Δ-groupoid G is freely generated by a quiver (oriented graph) consisting of two vertices A and B and two arrows x and y with

$$\mathrm{dom}(x) = \mathrm{cod}(x) = \mathrm{dom}(y) = A, \quad \mathrm{cod}(y) = B,$$

with the distinguished subset

$$H = \{x^{\pm 1}, x^{\pm 2}, y^{\pm 1}, (xy)^{\pm 1}\}, \quad j : x \mapsto x^{-1},\ x^2 \mapsto y,\ x^{-2} \mapsto xy.$$

One can show that

$$A'G \simeq B'G \simeq \mathbb{Z}[t, 3^{-1}]/(\Delta_{3_1}(t)), \quad \Delta_{3_1}(t) = t^2 - t + 1.$$

Example 12 (The figure-eight knot) The gluing rules are given by the list

$$(u|a, b, c, d),\ (v|c, d, a, b),\ (a|p, q, q),\ (b|p, p, q),\ (c|q, p, p),\ (d|q, q, p).$$

In this case, there are no non-trivial identifications in the corresponding Δ-groupoid G, and the ring $A'G$ is isomorphic to the ring S from Example 10, while

$$B'G \simeq \mathbb{Z}\langle u^{\pm 1}, v^{\pm 1}, w^{\pm 1} |\ u(u+1) = w,\ v(v+1) = w^{-1}, (uvu^{-1}v^{-1})^2 = w\rangle.$$

8 Homology of Δ-groupoids

Given a Δ-groupoid G with the distinguished generating subset H. We define recursively the following sequence of sets: $V_{-1} = \{*\}$ (a singleton or one element set), $V_0 = \pi_0(G)$ (the set of connected components of G), $V_1 = \mathrm{Ob}(G)$ (the set of objects or identities of G), $V_2 = H$, V_3 is the set of all H-composable pairs, while V_{n+1} for $n > 2$ is the collection of all n-tuples $(x_1, x_2, \dots, x_n) \in H^n$, satisfying the following conditions:

$$(x_i, x_{i+1}) \in V_3, \quad 1 \le i \le n - 1,$$

and

$$\partial_i(x_1, \dots, x_n) \in V_n, \quad 1 \le i \le n + 1,$$

where

$$\partial_i(x_1, \dots, x_n) = \begin{cases} (x_2, \dots, x_n), & i = 1; \\ (x_1, \dots, x_{i-2}, x_{i-1}x_i, x_{i+2}, \dots, x_n), & 2 \le i \le n; \\ (x_1, \dots, x_{n-1}), & i = n + 1. \end{cases} \tag{37}$$

Remark 5 If $N(G)$ is the nerve of G, then the system $\{V_n\}_{n\geq 1}$ can be defined as the maximal system of subsets $V_{n+1} \subset N(G)_n \cap H^n$, $n = 0, 1, \ldots$, (with the identification $H^0 = \mathrm{Ob}(H)$) closed under the face maps of the simplicial set $N(G)$.

For any H-composable pair (x, y) we introduce a binary operation

$$x * y = j(k(x)j(y)) = jk(x)j(xy) = ij(x)j(xy). \tag{38}$$

Lemma 8 *For any integer $n \geq 2$, if $(x_1, \ldots, x_n) \in V_{n+1}$, then the $(n-1)$-tuple*

$$\partial_0(x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1}), \quad y_i = z_i * x_{i+1}, \quad z_i = x_1 x_2 \cdots x_i, \tag{39}$$

is an element of V_n.

Proof We proceed by induction on n. For $n = 2$ the statement is evidently true. Choose an integer $k \geq 3$ and assume the statement is true for $n = k - 1$. Let us prove that it is also true for $n = k$. Taking into account the formula

$$y_i = ij(z_i)j(z_{i+1}), \quad 1 \leq i \leq k-1,$$

we see that for any $1 \leq i \leq k-2$, the pair (y_i, y_{i+1}) is H-composable with the product

$$y_i y_{i+1} = ij(z_i)j(z_{i+2}) = ij(z_i)j(z_i x_{i+1} x_{i+2}) = z_i * (x_{i+1} x_{i+2}).$$

Now, the $(k-2)$-tuples

$$\begin{aligned}&(y_1, \ldots, y_{i-1}, y_i y_{i+1}, y_{i+2}, \ldots, y_{k-1})\\ &\quad = \partial_0(x_1, \ldots, x_i, x_{i+1}x_{i+2}, x_{i+3}, \ldots, x_k), \quad 1 \leq i \leq k-2,\end{aligned}$$

as well as

$$(y_1, \ldots, y_{k-2}) = \partial_0(x_1, \ldots, x_{k-1}), \quad (y_2, \ldots, y_{k-1}) = \partial_0(x_1 x_2, \ldots, x_{k-1}),$$

are all in V_{k-1} by the induction hypothesis, and thus $(y_1, \ldots, y_{k-1}) \in V_k$. □

Definitions (37), (39) also make sense for $n = 2$, and, additionally, we extend them for three more values $n = -1, 0, 1$ as follows:

$(n = -1)$ as V_{-1} is a terminal object in the category of sets, there are no other choices but one for the map $\partial_0|_{V_0}$;

$(n = 0)$ if for $A \in V_1$ we denote $[A]$ the connected component of G defined by A, then

$$\partial_0 A = [A^*], \ \partial_1 A = [A]; \tag{40}$$

$(n = 1)$ using the domain (source) and the codomain (target) maps of the groupoid G (viewed as a category), we define

$$\partial_0 x = \mathrm{cod}(j(x)), \quad \partial_1 x = \mathrm{cod}(x), \quad \partial_2 x = \mathrm{dom}(x), \quad x \in V_2. \tag{41}$$

For any $n \geq -1$, let $B_n = \mathbb{Z}V_n$ be the abelian group freely generated by the elements of V_n. Define also $B_n = 0$, if $n < -1$. We extend linearly the maps ∂_i, $i \in \mathbb{Z}_{\geq 0}$ to a family of group endomorphisms

$$\partial_i : B \to B, \quad B = \oplus_{n\in\mathbb{Z}} B_n$$

so that $\partial_i|_{B_n} = 0$ if $i > n$. Then, the formal linear combination

$$\partial = \sum_{i\geq 0}(-1)^i \partial_i$$

is a well defined endomorphism of the group B such that $\partial B_n \subset B_{n-1}$.

Theorem 6 *The group B is a chain complex with differential ∂ and grading operator $p\colon B \to B, \quad p|_{B_n} = n$.*

Proof It is immediate to see that the group homomorphism $\partial' = \partial_0 - \partial$ is a restriction of the differential of the standard integral (augmented) chain complex of the nerve $N(G)$ so that $\partial'^2 = 0$. Now, due to the latter equation, the equation $\partial^2 = 0$ is equivalent to the equation

$$\partial_0^2 = \partial_0\partial' + \partial'\partial_0$$

which can straightforwardly be checked on basis elements. □

Lemma 9 *There exists a unique sequence of set-theoretical maps*

$$\delta_i : \mathbb{S}_\infty \to \mathbb{S}_\infty, \quad i \in \mathbb{Z}_{\geq 0},$$

such that for any $j \in \mathbb{Z}_{>0}$,

$$\delta_i(s_j) = \begin{cases} s_{j-1} & \text{if } i < j-1; \\ 1 & \text{if } i \in \{j-1, j\}; \\ s_j & \text{if } i > j, \end{cases} \tag{42}$$

and

$$\delta_i(gh) = \delta_i(g)\delta_{g^{-1}(i)}(h), \quad \forall g, h \in \mathbb{S}_\infty. \tag{43}$$

Proof From the identity

$$\delta_i(g) = \delta_i(1g) = \delta_i(1)\delta_i(g)$$

it follows immediately that $\delta_i(1) = 1$ for any $i \in \mathbb{Z}_{\geq 0}$. To prove the statement it is enough to check consistency of the defining relations of the group $\mathbb{S}_\infty$ with formulae (42), (43), i.e. the equations

$$\begin{aligned} \delta_i(s_j)\delta_{s_j(i)}(s_k) &= \delta_i(s_k)\delta_{s_k(i)}(s_j), \quad |j-k| > 1, \\ \delta_i(s_j)\delta_{s_j(i)}(s_{j+1})\delta_{s_{j+1}s_j(i)}(s_j) &= \delta_i(s_{j+1})\delta_{s_{j+1}(i)}(s_j)\delta_{s_js_{j+1}(i)}(s_{j+1}), \\ \delta_i(s_j)\delta_{s_j(i)}(s_j) &= 1, \quad \forall i \in \mathbb{Z}_{\geq 0}, \ \forall j \in \mathbb{Z}_{>0}. \end{aligned}$$

This is a straightforward verification. □

Remark 6 For any $n \geq 1$, we have $\delta_i(\mathbb{S}_n) \subset \mathbb{S}_{n-1}$, $0 \leq i \leq n$.

Lemma 10 *For any $n \geq -1$ the set V_n has a unique canonical (i.e. independent of particularities of a Δ-groupoid) structure of an $\mathbb{S}_{n+1}$-set such that*

$$\partial_i \circ g = \delta_i(g) \circ \partial_{g^{-1}(i)}, \quad 0 \leq i \leq n, \ g \in \mathbb{S}_{n+1}. \tag{44}$$

Proof For $n \leq 0$ the statement is trivial. For $n > 0$ and $g \in \{s_1, \dots, s_n\}$, Eqs. (44) take the form

$$\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i & \text{if } i < j-1; \\ \partial_j & \text{if } i = j-1; \\ \partial_{j-1} & \text{if } i = j; \\ s_j \circ \partial_i & \text{if } i > j. \end{cases} \tag{45}$$

In the case $n = 1$, Eqs. (40), (45) these imply that for any $A \in V_1$

$$\partial_i(s_1(A)) = \partial_i(A^*), \quad i \in \{0, 1\}.$$

The only canonical solution to these equations is of the form

$$s_1(A) = A^*, \quad \forall A \in V_1. \tag{46}$$

In the case $n = 2$, Eqs.(41), (45) and (46) imply that

$$\partial_m \circ s_1 = \partial_m \circ j, \quad \partial_m \circ s_2 = \partial_m \circ i, \quad 0 \leq m \leq 2,$$

with only one canonical solution of the form

$$s_1(x) = j(x), \quad s_2(x) = x^{-1}, \quad \forall x \in V_2.$$

In the case $n > 2$, one can show that the following formula constitutes a solution to system (45):

$$s_i(x_1, \dots, x_{n-1}) = \begin{cases} (j(x_1), x_1 * x_2, (x_1x_2) * x_3, \dots, (x_1 \cdots x_{n-2}) * x_{n-1}) & \text{if } i = 1; \\ (x_1^{-1}, x_1x_2, x_3, \dots, x_{n-1}) & \text{if } i = 2; \\ (x_1, \dots, x_{i-3}, x_{i-2}x_{i-1}, x_{i-1}^{-1}, x_{i-1}x_i, x_{i+1}, \dots, x_{n-1}) & \text{if } 2 < i < n; \\ (x_1, \dots, x_{n-3}, x_{n-2}x_{n-1}, x_{n-1}^{-1}) & \text{if } i = n, \end{cases}$$

where we use the binary operation (38). Let us show that there are no other (canonical) solutions.

We proceed by induction on n. The case $n \leq 2$ has already been proved. Assume, that the solution is unique for $n = m - 1 \geq 2$. For $n = m$, Eq. (44) with $i \in \{1, m\}$ implies that

$$\begin{aligned} (y_2, \dots, y_{m-1}) &= \delta_1(g)(\partial_{g^{-1}(1)}(x_1, \dots, x_{m-1})), \\ (y_1, \dots, y_{m-2}) &= \delta_m(g)(\partial_{g^{-1}(m)}(x_1, \dots, x_{m-1})), \end{aligned}$$

where $(y_1, \dots, y_{m-1}) = g(x_1, \dots, x_{m-1})$. By the induction hypothesis, the right hand sides of these equations are uniquely defined, and so are their left hand sides. The latter, in turn, uniquely determine the $(m-1)$-tuple $(y_1, \dots, y_{m-1})$, and, thus, the solution is unique for $n = m$. □

Theorem 7 *The sub-group $A \subset B$, generated by the set of elements*

$$\cup_{n \geq 1} \{x + s_i x \mid x \in V_n, \ 1 \leq i \leq n\},$$

is a chain sub-complex so that there is a short exact sequence of chain complexes:

$$0 \to A \to B \to C = B/A \to 0.$$

Proof From the definition of A, it follows that $A = \oplus_{n\in\mathbb{Z}} A_n$ with $A_n = A \cap B_n$. Let us show that $\partial A_n \subset A_{n-1}$. For any $x \in V_n$ and $1 \le i \le n$, we have

$$\begin{aligned}\partial(x + s_i x) &= \sum_{j=0}^{i-2}(-1)^j \partial_j(x + s_i x) + \sum_{j=i-1}^{i}(-1)^j \partial_j(x + s_i x) + \sum_{j=i+1}^{n}(-1)^j \partial_j(x + s_i x)\\ &= \sum_{j=0}^{i-2}(-1)^j (1+s_{i-1})\partial_j x + \sum_{j=i-1}^{i}(-1)^j (\partial_j x+\partial_{s_i(j)} x) + \sum_{j=i+1}^{n}(-1)^j (1+s_i)\partial_j x\\ &= \sum_{j=0}^{i-2}(-1)^j (1 + s_{i-1})\partial_j x + \sum_{j=i+1}^{n}(-1)^j (1 + s_i)\partial_j x \in A_{n-1}.\end{aligned}$$

□

Definition 4 *Integral homology* $H_*(G)$ of a Δ-groupoid G is the homology of the chain complex $C = A/B$.

Conjecture 1 Let T and T' be two ideal triangulations of a (cusped) 3-manifold, related by a $2-3$ Pachner move. Let G and G' be the associated Δ-groupoids. Then, the corresponding homology groups $H_*(G)$ and $H_*(G')$ are isomorphic.

Remark 7 Conjecture 1 is true for any hyperbolic knot K when the ideal triangulations are geometric. In this case the corresponding Δ-groupoid homology is not interesting: one has the isomorphism $H_*(G_K) \simeq \tilde{H}_*(\mathbb{S}^3/K, \mathbb{Z})$, and it is well known fact that the homology of the space $\mathbb{S}^3/K$ is independent of K:

$$\tilde{H}_n(\mathbb{S}^3/K, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n \in \{2, 3\};\\ 0, & \text{otherwise.}\end{cases}$$

However, it might be interesting for non-hyperbolic knots, for example, one has the following results of calculations for the two simplest torus knots of types (2, 3) and (2, 5) (with some particular choices of ideal triangulations):

$$H_n(G_{3_1}) = \begin{cases} \mathbb{Z}, & \text{if } n = 2;\\ 0, & \text{otherwise.}\end{cases} \quad H_n(G_{5_1}) = \begin{cases} \mathbb{Z}, & \text{if } n = 2;\\ \mathbb{Z}_5, & \text{if } n = 3;\\ 0, & \text{otherwise.}\end{cases}$$

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